Key

Birzeit University
Mathematics Department
Math332
Second Exam

Instructor: Dr. Ala Talahmeh
Name:...
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Exercise#1 [5+5 points].

(a) Solve the following PDE

$$u_{yx} = 2\cos y + \frac{1}{\sqrt{1-x^2}}, \quad u_y(1,y) = \frac{\pi}{2}, \quad u(x,0) = \sin x.$$

(b) Find a solution in **exponential form** for the following PDE

$$u_{xx} - u_{yy} - 2u_x + 4u_y - 3u = 0.$$

(a) Integrate both sides w.r. to x,

$$u_y = 2x\cos y + \sin^{-1}x + f(y) \quad 1$$

$$u_y(1,y) = 2\cos y + \frac{\pi}{2} + f(y) = \frac{\pi}{2}$$

$$\Rightarrow \boxed{f(y) = -2\cos y} \quad 1$$

$$\therefore u_y = 2(x-1)\cos y + \sin^{-1}x$$

Integrate both sides w.r. to y,

$$u(x,y) = 2(x-1)\sin y + y\sin^{-1}x + g(x) \quad 1$$

$$u(x,0) = g(x) = \sin x \quad 1$$

$$\therefore \boxed{u(x,y) = 2(x-1)\sin y + y\sin^{-1}x + \sin x} \quad 1$$

[Cont......]

(b) let $u(x,y) = e^{\alpha x + \beta y}$ (1). then the aux. eq. is $\alpha^2 - \beta^2 - 2\alpha + 4\beta - 3 = 0$ (1)

$$\Rightarrow (\alpha - 1)^2 - (\beta - 2)^2 = 0 \quad (1)$$

$$\Rightarrow \alpha = 1 \pm (\beta - 2)$$

$$\boxed{\alpha = \beta - 1} \quad \text{or} \quad \boxed{\alpha = 3 - \beta} \quad (1)$$

$$\therefore u(x,y) = c_1 e^{(\beta-1)x + \beta y} + c_2 e^{(3-\beta)x + \beta y} \quad (1)$$

Exercise#2 [5+5 points].

(a) Determine the regions in the xy−plane for which the following PDE

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x} + 2x\sin y\,\frac{\partial u}{\partial y}\right) + \frac{\partial^2 u}{\partial y^2} = 5u$$

is **hyperbolic**, **parabolic**, or **elliptic**.

(b) Find the constants A_n such that

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh ny \sin nx$$

is the solution of the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi,\ 0 < y < \pi, \\ u(0,y) = 0,\ u(\pi,y) = 0, & 0 < y < \pi, \\ u(x,0) = 0,\ u(x,\pi) = 1, & 0 < x < \pi. \end{cases}$$

1

(a) $x^2 u_{xx} + 2x u_x + 2\sin y\, u_y + 2x\sin y\, u_{xy} + u_{yy} = 5u$

$\Delta = B^2 - 4AC$ 2

$= 4x^2\sin^2 y - 4x^2 \cdot 1 = -(2x\cos y)^2$

the eq. is parabolic if $\boxed{x=0}$ or 1

$\boxed{y = (2n+1)\frac{\pi}{2}}$, $n = 0, \pm1, \pm2, \ldots$

Elliptic if $x \neq 0$ and 1

$y \neq (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \ldots$

[Cont......]

2

(b) $$u(x,\pi) = \sum_{n=1}^{\infty} A_n \sinh n\pi \, \sin nx = 1$$

$$\Rightarrow A_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} 1 . \sin nx \, dx \quad 1$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_0^{\pi}$$

$$= \frac{2(1-(-1)^n)}{n\pi}$$

2

$$\therefore A_n = \frac{2(1-(-1)^n)}{n\pi \sinh n\pi}$$

Exercise#3 [6+4 points].

(a) Represent $f(x) = e^{-2x}$, $x > 0$ by Fourier **cosine** integral.

(b) Use the result in **(a)** to show that

$$\int_0^\infty \frac{\cos 2x}{1+x^2}\, dx = \frac{\pi}{2e^2}.$$

1

1(a) $f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x \, d\alpha$, where

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x \, dx$$

$$= \int_0^\infty e^{-2x} \cdot \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}\, dx \quad 2$$

$$= \int_0^\infty \frac{e^{-(2-i\alpha)x} + e^{-(2+i\alpha)x}}{2}\, dx \quad 1$$

$$= \frac{1}{2}\left[-\frac{e^{-(2-i\alpha)x}}{2-i\alpha} - \frac{e^{-(2+i\alpha)x}}{2+i\alpha}\right]_0^\infty$$

1

$$= \frac{1}{2}\left[\frac{1}{2-i\alpha} + \frac{1}{2+i\alpha}\right] = \frac{2}{4+\alpha^2}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{2}{4+\alpha^2}\cos\alpha x \, d\alpha$$

1

or $$\boxed{e^{-2x} = \frac{2}{\pi}\int_0^\infty \frac{\cos\alpha x}{4+\alpha^2}\, d\alpha}$$

(5)

[Cont......]

(b) put $x=1$, $e^{-2} = \frac{4}{\pi}\int_0^\infty \frac{\cos\alpha}{4+\alpha^2}\,d\alpha$ (1)

or $\int_0^\infty \frac{\cos\alpha}{4+\alpha^2}\,d\alpha = \frac{\pi}{4e^2}$ (1)

Let $\alpha = 2x \Rightarrow d\alpha = 2dx$ (1)

$\Rightarrow \int_0^\infty \frac{\cos 2x}{4+4x^2}\cdot 2\,dx = \frac{\pi}{4e^2}$ (1)

or $\int_0^\infty \frac{\cos 2x}{1+x^2} = \frac{\pi}{2e^2}$ ∎

Exercise#4 [6+4 points].

(a) Use **D'Alembert's formula** to show that the solution of the IVP

$$\begin{cases} u_{xx} = u_{tt}, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = x^2, & u_t(x,0) = \frac{x^2}{x^2+1}. \end{cases}$$

is

$$u(x,t) = x^2 + t^2 + t + \frac{\tan^{-1}(x-t) - \tan^{-1}(x+t)}{2}.$$

(b) Consider the following IVP

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x), & u_t(x,0) = g(x). \end{cases}$$

Show that if both $f(x)$ and $g(x)$ are even, then $u_x(0,t) = 0$.
Hint. Use **D'Alembert's formula**.

(a) $u(x,t) = \dfrac{f(x-t) + f(x+t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} g(s)\,ds$ 1

where $f(x) = x^2$, $g(x) = \dfrac{x^2}{x^2+1}$.

$\therefore u(x,t) = \dfrac{(x-t)^2 + (x+t)^2}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \dfrac{s^2}{s^2+1}\,ds$ 1

$= \dfrac{x^2 - 2xt + t^2 + x^2 + 2xt + t^2}{2}$ 2

$+ \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \left(1 - \dfrac{1}{s^2+1}\right) ds$

$= x^2 + t^2 + \dfrac{1}{2}\left[s - \tan^{-1} s \right]_{x-t}^{x+t}$ 1

$= x^2 + t^2 + t + \dfrac{\tan^{-1}(x-t) - \tan^{-1}(x+t)}{2}$ 1

[Cont......]

$$\text{(b)}\quad u(x,t) = \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$ 1

$$u_x(x,t) = \frac{f'(x-at)+f'(x+at)}{2}$$ 1

$$+\frac{1}{2a}\Big[g(x+at)-g(x-at)\Big]$$

1

$$u_x(0,t) = \frac{f'(-at)+f'(at)}{2} + \frac{1}{2a}\big(g(at)-g(-at)\big)$$

Now, since f is even, then f' is odd

so, $f'(-at) = -f'(at)$. 1

g is even, then $g(at) = g(-at)$.

$$\therefore\ u_x(0,t) = \frac{-f'(at)+f'(at)}{2} + \frac{1}{2a}\big(g(at)-g(at)\big)$$

$$= 0$$

Exercise#5. [15 points]. Use **seperation of variables method** to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0,$$

subject to the boundary conditions

$$\begin{cases} u(0,y) = 0, \quad u(\pi,y) = 0, \quad y > 0, \\ u(x,0) = \sin 2x. \end{cases}$$

where $u(x,y)$ is **bounded** as $y \to +\infty$.

let $u = XY \Rightarrow X''Y = -XY''$ (1)

$\Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$

$\Rightarrow \boxed{X'' + \lambda X = 0}, \quad \boxed{Y'' - \lambda Y = 0}$ (1)

$u(0,y) = 0 \Rightarrow X(0) = 0, \quad u(\pi,y) = 0 \Rightarrow X(\pi) = 0$

We consider the eigenvalue problem $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 = X(\pi) \end{cases}$

Case I. $\lambda = 0, \quad X = C_1 + C_2 x$

$X(0) = C_1 = 0, \quad X(\pi) = \pi C_2 = 0 \Rightarrow C_2 = 0$ (1)

$\Rightarrow$ trivial solution.

Case II. $\lambda = -\alpha^2 < 0, \quad \alpha > 0$

(1) $X = C_3 \cosh \alpha x + C_4 \sinh \alpha x$

$X(0) = C_3 = 0$

(9)

$X(\pi) = C_4 \underbrace{\sinh(\alpha\pi)}_{\neq 0} = 0 \Rightarrow C_4 = 0$ trivial Solution

[Cont......]

Case III. $\lambda = \alpha^2 > 0$, $\alpha > 0$ 2

$$X = C_5 \cos\alpha x + C_6 \sin\alpha x$$

$$X(0) = C_5 = 0,$$

$$X(\pi) = C_6 \sin\alpha\pi = 0 \text{. Let } C_6 \neq 0, \sin\alpha\pi = 0$$

$$\Rightarrow \alpha\pi = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \boxed{\alpha_n = n}, \quad \boxed{X_n = C_6 \sin nx}$$

Now, for $\lambda_n = \alpha_n^2 = n^2$, $Y'' - n^2 Y = 0$ 1

$$\Rightarrow Y_n = C_7 e^{ny} + C_8 e^{-ny} \quad (\text{since } u \text{ is bdd})$$

1 since $e^{ny} \to \infty$ as $y \to \infty$, $e^{-ny} \to 0$ as $y \to \infty$ and $u(x,y)$ is bdd as $y \to \infty$, then we must put $C_7 = 0$

1 $\therefore \boxed{Y_n = C_8 e^{-ny}}$

1 $\therefore u(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx$, where $A_n = C_6 C_8$.

1 $u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin 2x$

2 $\Rightarrow A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots = \sin 2x$

$$\Rightarrow \boxed{A_2 = 1}, \quad A_n = 0, \ \forall n \neq 2$$

2 $\therefore u(x,y) = e^{-2y} \sin 2x$.

(10)

Exercise#6 [15 points]. Use **seperation of variables method** to solve the heat problem

$$5\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \ t > 0,$$

subject to the boundary and initial conditions

$$\begin{cases} u_x(0,t) = 0, \ u_x(\pi,t) = 0, \ t > 0, \\ u(x,0) = x, \ 0 < x < \pi. \end{cases}$$

let $u(x,t) = X(x)\,T(t)$ (1)

$\Rightarrow 5X''T = XT'$

$\Rightarrow \dfrac{X''}{X} = \dfrac{T'}{5T} = -\lambda$ (1)

$\boxed{X'' + \lambda X = 0}$, $\boxed{T' + 5\lambda T = 0}$

$u_x(0,t) = 0 \Rightarrow X'(0) = 0$

$u_x(\pi,t) = 0 \Rightarrow X'(\pi) = 0$

Case I. $\lambda = 0$, $X(x) = c_1 + c_2 x$

$X' = c_2$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(\pi) = 0 \Rightarrow c_2 = 0$ $\Rightarrow X = c_1 \neq 0$ (1)

nontrivial solution

Case II. $\lambda = -\alpha^2 < 0$, $\alpha > 0$

$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

(1)

$X' = \alpha c_1 \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$$X'(\pi) = 0 \Rightarrow c_1 \alpha \underbrace{\sinh \alpha\pi}_{\neq 0} = 0 \Rightarrow c_1 = 0$$

[Cont...QV]

Trivial solution 1

Case III. $\lambda = \alpha^2 > 0, \ \alpha > 0$ 2

$$X = c_1 \cos\alpha x + c_2 \sin\alpha x$$

$$X' = -\alpha c_1 \sin\alpha x + \alpha c_2 \cos\alpha x$$

$$X'(0) = \alpha c_2 = 0 \Rightarrow c_2 = 0$$

$$X'(\pi) = 0 \Rightarrow -c_1 \alpha \sin\alpha\pi = 0$$

Let $c_1 \neq 0$, $\sin\alpha\pi = 0 \Rightarrow \alpha\pi = n\pi \Rightarrow \alpha_n = n, \ n = 1, 2, \ldots$

$$X(x) = c_1 \cos nx$$

Now, for $\lambda = 0$, $T(t) = c_3$ constant

For $\lambda = n^2$, $\boxed{T_n(t) = c_4 e^{-5n^2t}}$ 2

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-5n^2t} \cos nx}$$ 1

$$u(x,0) = x = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$ 1

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \pi/2$$ 1

$$A_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2\left((-1)^n - 1\right)}{n^2\pi}.$$ 2

Good Luck

(12)

$$\therefore u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n - 1\right)}{n^2\pi} e^{-5n^2t} \cos nx.$$ 2